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Article

Design and analysis of a new iterative family for solving nonlinear equations

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Abstract: This article addresses the problem of improving the convergence order and stability of iterative methods for solving nonlinear equations. The main objective is to design a new multipoint iterative family with sixth-order convergence and to analyze both its convergence behavior and complex dynamics. The methodology combines the theoretical analysis of the convergence order, the derivation of the associated rational operator, and the use of complex dynamics tools such as stability surfaces, parameter planes, and dynamical planes. Numerical experiments conducted on nonlinear test equations confirm the results obtained from the convergence and stability analysis. The proposed method achieves high accuracy in few iterations, maintaining the Approximate Computational Order of Convergence (ACOC) around six and exhibiting competitive efficiency compared to classical methods such as Newton, Ostrowski, Jarratt, and CMT. The conclusions highlight the robustness of the family with respect to initial conditions. The findings have theoretical implications for the design of high-order iterative methods and practical implications for solving scientific and engineering problems more efficiently.

Keywords: complex dynamic, multi-step iterative methods, nonlinear equations, stability.

1. Introduction

To solve problems in science and engineering, it is common to formulate a nonlinear equation or a system of nonlinear equations, depending on the number of variables or the specific problem conditions. This type of equation arises in chemical processes, astronomical applications, polynomial interpolation, and the discretization of one-dimensional boundary value problems, among others. However, due to their nonlinear nature, obtaining exact solutions through traditional algebraic techniques is often highly complex or even unfeasible. In such cases, it becomes necessary to resort to iterative methods, which allow the solution to be approximated from an initial estimate and generate a sequence of values that, under certain conditions, may converge to the exact solution.

The resolution of nonlinear equations and systems by iterative methods has been, and continues to be, a recurring topic in numerical analysis, as evidenced by numerous publications (Artidiello, 2014; Moscoso-Martínez et al., 2023; Ortega & Rheinboldt, 1970; Traub, 1964), among others. One of the most well-known iterative methods for solving a nonlinear equation $f(x) = 0$ is Newton's method, which is given by the following expression:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots$$

Nevertheless, this method exhibits limitations that motivate the search for new schemes with improved convergence properties.

The present research proposes a multipoint iterative family of order six for solving nonlinear equations. Because the proposed family depends on a parameter, the method's behavior may vary significantly with its value. For this reason, a convergence analysis is carried out to determine the method's order as a function of this parameter. Furthermore, a dynamical analysis is incorporated to identify the most stable and efficient elements of the family, thereby facilitating its application in numerical tests with nonlinear functions of greater complexity.

2. Literature Review

The resolution of nonlinear and linear systems of equations is a central problem in numerical analysis, with applications in physics and chemistry, boundary value problems, and polynomial interpolation. Since obtaining exact solutions is often intractable, numerous iterative methods have been developed with different structures and convergence orders.

In recent literature, several multipoint schemes of varying order have been introduced. Singh (2018) designed a Steffensen-type method of order five for nonlinear systems, while Chicharro et al. (2019) proposed a biparametric family of order six, combining Newton's and Traub's methods. More recently, Cordero et al. (2024) introduced an innovative technique that increases the order of convergence by three without relying on Jacobian matrices, demonstrating strong efficiency in biological models.

The pursuit of higher orders has motivated the development of more advanced schemes. Shams et al. (2022) constructed a three-step family with order eight, and Wang (2021) also proposed a multipoint method for nonlinear systems, achieving order eight. On the other hand, Tao & Madhu (2019) designed a scheme with order sixteen, although at a high computational cost. It is worth noting, however, that not all these studies include complex dynamics or stability analysis. For instance, Wang (2021), despite achieving order eight, did not address stability, thereby limiting understanding of the scheme's global behavior.

In recent years, parametrized iterative families have been developed that integrate tools from fundamental and complex dynamics to characterize the stability of fixed points and the structure of basins of attraction (Khirallah & Alkhomsan, 2022). This work highlights that dynamic analysis is an essential complement to classical convergence studies, as it provides insights into stability regions and chaotic behaviors.

From this overview, it is evident that most existing methods exhibit convergence orders between three and six. Moreover, complete dynamical studies are not always included. Consequently, there is a clear research gap for designing schemes that combine high convergence order with a comprehensive stability analysis. The iterative family proposed in this work addresses this gap by introducing a sixth-order multipoint scheme supported by theoretical convergence analysis and complex dynamical study.

3. Material and Methods

The research methodology adopted in this work is both theoretical and applied. A new step within an iterative family is proposed to increase the order of convergence without significantly raising the computational cost. In the theoretical phase, concepts and theorems from numerical analysis will be employed to rigorously establish the order of convergence of the new scheme. Likewise, specific conditions on the parameter under consideration will be analyzed to simplify the method's general expression.

The theoretical order of convergence will be contrasted with the approximate computational order of convergence (ACOC), following the proposal of Cordero and Torregrosa (2007). To analyze the method's dynamic behavior, a wide range of initial conditions will be considered. The corresponding orbits will be constructed from these, and their final states will be identified. This analysis will allow the generation of basins of attraction, which illustrate the regions of the plane where the method exhibits greater stability. A predefined error tolerance and a maximum number of iterations will define the visualization criterion. In addition, the effects of the parameter on the method's stability and rate of convergence, as well as its interactions with the different fixed points, will be examined.

The applied research will corroborate theoretical results. For this purpose, the scheme will be applied to low-order nonlinear systems and to systems with exponential, logarithmic, and trigonometric equations. MATLAB will be used for computational calculations to verify the theoretical results obtained. Both MATLAB and Mathematica will be employed to generate graphs.

4. Results

4.1 Development of the New Iterative Family

The starting point is the fourth-order iterative method proposed by Artidiello (2014), whose expression is

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots, \\ x_{k+1} = y_k - \left(1 + 2\dfrac{f(y_k)}{f(x_k)}\right)\dfrac{f(y_k)}{f'(x_k)} \end{cases}.$$

It is intended to improve this method; for this purpose, a third step is proposed in Moscoso-Martínez et al. (2023). By incorporating this third step into the method, a new triparametric iterative scheme is obtained.

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots, \\ z_k = y_k - (1 + 2s_k)\dfrac{f(y_k)}{f'(x_k)}, \\ x_{k+1} = z_k - (\alpha + \beta s_k + \gamma t_k)\dfrac{f(z_k)}{f'(x_k)}, \end{cases}$$

where $\boldsymbol{s_k} = \frac{f(y_k)}{f(x_k)}$ and $\boldsymbol{t_k} = \frac{f'(x_k)}{f[x_k,y_k]}$, where $\boldsymbol{f[x_k, y_k]}$ is the divided difference operator defined in (Ortega & Rheinboldt, 1970), with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ being arbitrary parameters.

4.2 Convergence Analysis of the New Family

This section presents the convergence analysis of the new triparametric iterative family. Furthermore, a strategy is proposed to reduce the triparametric scheme to a uniparametric one to accelerate convergence.

Theorem 1. *Let $f: C \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function on an open and connected set C, and let $\xi \in C$ be a root of $f(x)$ such that f' is continuous at ξ and $f'(\xi) \neq 0$. Then, if x_0 is an initial guess sufficiently close to ξ the members of the family converge to ξ with order of convergence four for the error equation being*

$$e_{k+1} = -(c_2(5c_2^2 - c_3)(\alpha + \gamma - 1))e_k^4 + O(e_k^5),$$

where $e_k = x_k - \xi$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\xi)}{f'(\xi)}$, for $j = 2,3,\cdots$

Proof. Expanding $f(x_k)$ and $f'(x_k)$ around ξ, we obtain

$$\begin{aligned} f(x_k) &= f(\xi) + f'(\xi)e_k + \frac{1}{2!}f''(\xi)e_k^2 + \cdots + \frac{1}{6!}f^{(vi)}(\xi)e_k^6 + O(e_k^7) \\ &= f'(\xi)\left(e_k + \frac{1}{2!}\frac{f''(\xi)}{f'(\xi)}e_k^2 + \cdots + \frac{1}{6!}\frac{f^{(vi)}(\xi)}{f'(\xi)}e_k^6 + O(e_k^7)\right) \\ &= f'(\xi)\left(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + c_5e_k^5 + c_6e_k^6 + O(e_k^7)\right). \end{aligned}$$

$$\begin{aligned} f'(x_k) &= f'(\xi) + f''(\xi)e_k + \frac{1}{2!}f'''(\xi)e_k^2 + \cdots + \frac{1}{6!}f^{(v)}(\xi)e_k^5 + O(e_k^6) \\ &= f'(\xi)\left(1 + \frac{f''(\xi)}{f'(\xi)}e_k + \frac{1}{2!}\frac{f'''(\xi)}{f'(\xi)}e_k^2 + \cdots + \frac{1}{6!}\frac{f^{(v)}(\xi)}{f'(\xi)}e_k^5 + O(e_k^6)\right) \\ &= f'(\xi)(1 + 2c_2e_k + 3c_3e_k^2 + 4c_4e_k^3 + 5c_5e_k^4 + 6c_6e_k^5 + O(e_k^6)), \end{aligned}$$

where $c_j = \frac{1}{j!}\frac{f^{(j)}(\xi)}{f'(\xi)}$, $j = 2,3,\cdots$

Therefore, using $f(x_k)$ and $f'(x_k)$, the first step of the family is:

$$\begin{aligned}y_k-\xi &= c_2e_k^2+(-2c_2^2+2c_3)e_k^3+(4c_2^3-7c_2c_3+3c_4)e_k^4\\&\quad+(-8c_2^4+20c_2^2c_3-6c_3^2-10c_2c_4+4c_5)e_k^5+(16c_2^5-52c_2^3c_3\\&\quad+28c_2^2c_4-17c_3c_4+c_2(33c_3^2-13c_5)+5c_6)e_k^6+O(e_k^7).\end{aligned}$$

Using the expression of y_k the expansion corresponding to $f(y_k)$ is:

$$\begin{aligned}f(y_k) &= f'(\xi)(c_2e_k^2+2(-c_2^2+c_3)e_k^3+(5c_2^3-7c_2c_3+3c_4)e_k^4-2(6c_2^4\\&\quad-12c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e_k^5+(28c_2^5-73c_2^3c_3+34c_2^2c_4\\&\quad-17c_3c_4+c_2(37c_3^2-13c_5)+5c_6)e_k^6+O(e_k^7)).\end{aligned}$$

This implies that, with $f(x_k)$, $f'(x_k)$, and $f(y_k)$, the second step of the family is given by

$$\begin{aligned}z_k-\xi &= (5c_2^3-c_2c_3)e_k^4-2(18c_2^4-16c_3c_2^2+c_4c_2+c_3^2)e_k^5\\&\quad+(170c_2^5-262c_3c_2^3+48c_4c_2^2+(66c_3^2-3c_5)c_2-7c_3c_4)e_k^6+O(e_k^7).\end{aligned}$$

The expansion of $f(z_k)$ is

$$\begin{aligned}f(z_k) &= f'(\xi)\Big((5c_2^3-c_2c_3)e_k^4-2(18c_2^4-16c_3c_2^2+c_4c_2+c_3^2)e_k^5\\&\quad+(170c_2^5-262c_3c_2^3+48c_4c_2^2+(66c_3^2-3c_5)c_2-7c_3c_4)e_k^6+O(e_k^7)).\end{aligned}$$

The error equation obtained by using $f(x_k)$, $f'(x_k)$, $f(y_k)$, and $f(z_k)$ is

$$\begin{aligned}e_{k+1} &= -(c_2(5c_2^2-c_3)(\alpha+\gamma-1))e_k^4+(c_2^4(46\alpha-5\beta+41\gamma-36)\\&\quad+c_3c_2^2(-34\alpha+\beta-33\gamma+32)+2c_3^2(\alpha+\gamma-1)+2c_2c_4(\alpha+\gamma-1))e_k^5\\&\quad+(c_2^5(-262\alpha+61\beta-206\gamma+170)+c_3c_2^3(345\alpha-47\beta+299\gamma-262)\\&\quad+2c_4c_2^2(-26\alpha+\beta-25\gamma+24)+7c_3c_4(\alpha+\gamma-1)\\&\quad+c_2(c_3^2(-73\alpha+4\beta-69\gamma+66)+3c_5(\alpha+\gamma-1))\big)e_k^6+O(e_k^7).\end{aligned}$$

Which completes the proof.

Theorem 2. *Let $f: C \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function on an open and connected set C, and let $\xi \in C$ be a root of $f(x)$ such that f' is continuous at ξ and $f'(\xi) \neq 0$. Then, if x_0 is an initial guess sufficiently close to ξ the members of the family converge to ξ with an order of convergence of six whenever $\beta = 1+\alpha$ and $\gamma = 1-\alpha$ with the error equation being*

$$e_{k+1} = c_2(5c_2^2-c_3)\big((5+\alpha)c_2^2-c_3\big)e_k^6+O(e_k^7).$$

Proof. It is possible to increase the order of convergence of the scheme; to do so, the terms e_k^4 and e_k^5 In the error expression, the expression must be canceled. Therefore, the parameters must satisfy the following system of equations.

$$\begin{cases}\alpha+\gamma=1,\\ 46\alpha-5\beta+41\,\gamma=36,\\ -34\alpha+\beta-33\,\gamma=-32.\end{cases}$$

By manipulating the equations, the following expressions are obtained:

$$\beta = 1+\alpha \text{ and } \gamma = 1-\alpha,$$

and substituting the parameters, the new error equation is obtained

$$e_{k+1} = c_2(5c_2^2 - c_3)\big((5+\alpha)c_2^2 - c_3\big)e_k^6 + O(e_k^7).$$

Which proves the statement. □

From Theorem 2, it follows that if only the parameter α is kept, the new three-parameter iterative family reduces to a one-parameter family with order of convergence six, for any real or complex values of the parameters, provided that the condition is satisfied. The previous analysis has been developed assuming that ξ is a simple root. However, the family can be extended to the case of multiple roots by relying on the modified Newton's method for roots with multiplicity m , whose expression is given by

$$x_{k+1} = x_k - m\,\frac{f(x_k)}{f'(x_k)},$$

so that the order of convergence can be preserved without requiring significant modifications to the expression of the proposed family.

4.3 Stability analysis

This section focuses on the study of the dynamic properties of the rational operator determined by the iterative family. The dynamic analysis provides important information on the stability of each family member, depending on the initial approximations used. A rational operator will be obtained to conduct the analysis, thereby enabling the construction of parameter spaces and dynamical planes. These representations in the complex plane allow us to observe the behavior of the method as a function of the parameter $\boldsymbol{\alpha}$ and to study the basins of attraction, fixed points, and attractors. The rational operator of the family can be constructed for any nonlinear function $\boldsymbol{f(x)}$. In this case, it will be built from a quadratic polynomial, and the results obtained will then be extrapolated to functions of greater complexity.

Theorem 3. *Let $p(x) = (x-a)(x-b)$ be a generic polynomial with roots $a, b \in \mathbb{R}$. Then, the rational operator $O_\alpha(z)$ associated with the family and applied to $p(x)$ is*

$$O_\alpha(z) = \frac{z^6(z^2+4z+5)M(z)}{(5z^2+4z+1)N(z)}$$

where $\alpha \in \mathbb{C}$ is an arbitrary parameter, and moreover $M(z)$ and $N(z)$ are polynomials depending on the parameter $\alpha \in \mathbb{C}$.

Proof. We consider the generic polynomial, $\boldsymbol{p(x) = (x-a)(x-b)}$, where $\boldsymbol{a, b} \in \mathbb{R}$ are its roots. By applying $\boldsymbol{p(x)}$ to the family, we obtain a rational operator $\boldsymbol{R_p(x, \alpha, a, b)}$ which depends on the roots of the polynomial $\boldsymbol{p(x)}$. To eliminate the dependence on the roots, we consider the Möbius transformation defined as

$$\boldsymbol{H(z) = \frac{z-a}{z-b}},$$

which satisfies $\boldsymbol{H(\infty) = 1}$, $\boldsymbol{H(a) = 0}$ and $\boldsymbol{H(b) = \infty}$. Therefore, we obtain

$$\boldsymbol{O_\alpha(z) = (H \circ R \circ H^{-1})(z) = \frac{z^6(z^2+4z+5)M(z)}{(5z^2+4z+1)N(z)}},$$

where the polynomials have been defined as

$$\begin{aligned}\boldsymbol{M_\alpha(z)} &= \boldsymbol{\alpha + 5 + (34+4\alpha)z + (98+5\alpha)z^2 + 153z^3 + 149z^4}\\ &\quad \boldsymbol{+92z^5 + 37z^6 + 9z^7 + z^8},\\ \boldsymbol{N_\alpha(z)} &= \boldsymbol{1 + 9z + 37z^2 + 92z^3 + 149z^4 + 153z^5 + (98+5\alpha)z^6}\\ &\quad \boldsymbol{+(34+4\alpha)z^7 + (5+\alpha)z^8}.\end{aligned}$$

Since the factor $\boldsymbol{z^6}$ is involved in the operator $\boldsymbol{O_\alpha(z)}$, it is confirmed that the iterative family has an order of at least six in the case of quadratic equations.

Once the rational operator $\boldsymbol{O_\alpha(z)}$ has been obtained, it is possible to determine its fixed points and to classify them by stability.

Proposition 1. *Considering the equation $O_\alpha(z) = z$ the results obtained are:*

- $x = 0$ *and* $x = \infty$ *are fixed points of* $O_\alpha(x)$ *for any* $\alpha \in \mathbb{C}$.
- $x = 1$ *is a strange, fixed point.*
- *The roots of the polynomial*

$$k_\alpha(t) = 1 + 14t + 92t^2 + 377t^3 + 1079t^4 + (2263 - 5\alpha)t^5 + (3528 - 24\alpha)t^6 \\ +(4088 - 42\alpha)t^7 + (3528 - 24\alpha)t^8 + (2263 - 5\alpha)t^9 + 1079t^{10} \\ +377t^{11} + 92t^{12} + 14t^{13} + t^{14}$$

denoted by $ex_i(\alpha)$ *with* $i = 1,2,\cdots,14$, *are strange, fixed points of the operator* $O_\alpha(z)$.

To analyze the stability of the fixed points, one obtains

$$O'_\alpha(x) = \frac{-4x^6(1+x)^{10}(1+x^2)(1+x+x^2)Q(x)}{(-1-3x-4x^2-3x^3+(-1+2\alpha)x^4)^2(N(x))^2},$$

where has the polynom

$$Q(x) = -7(1+x)^4(1+x+x^2)^3(5+4x+5x^2) + 2\alpha(1+x)^2(1+x+x^2) \\ +2\alpha(1+x)^2(1+x+x^2)(35+46x+53x^2-22x^3-20x^4-22x^5 \\ +53x^6+46x^7+35x^8) + 2\alpha^2x(4+25x+72x^2+20x^3+2x^4-66x^5 \\ +2x^6+20x^7+72x^8+25x^9+4x^{10}).$$

Proposition 2. *The points* $x = 0$ *and* $x = \infty$ *are superattractors for all* $\alpha \in \mathbb{C}$. *The extraneous fixed point* $x = 1$ *satisfies the following:*

- *If* $|1445 + 25\alpha| > 6144$, *then* $x = 1$ *is an attractor. Moreover,* $x = 1$ *cannot be a superattractor.*
- *If* $|1445 + 25\alpha| < 6144$, *then* $x = 1$ *is a repeller.*
- *If* $|1445 + 25\alpha| = 6144$, *then* $x = 1$ *is parabolic.*

Let R_p be a rational operator determined by an iterative family, and let z_0 be a fixed point of R_p. Let $\alpha \in \mathbb{C}$ be a parameter of the iterative family. Then, the function $h: \mathbb{C} \to \mathbb{C}$ defined by $h(\alpha) = R'_p(z_0, \alpha)$ is called the stability function.

To numerically analyze the stability function, stability diagrams are used. These diagrams represent the surface generated in space by the function $\phi(\alpha) = |R'_p(z_0, \alpha)|$. On the Z-axis, the values of $|R'_p(z_0, \alpha)|$ are plotted, while on the XY-plane, the real and imaginary parts of $\alpha \in \mathbb{C}$ are represented. Through these visualizations, one can identify the regions where z_0 acts as a repulsive or attractive strange fixed point. The regions where the strange fixed point is repulsive are colored in gray, and those where it is attractive are shown in another color.

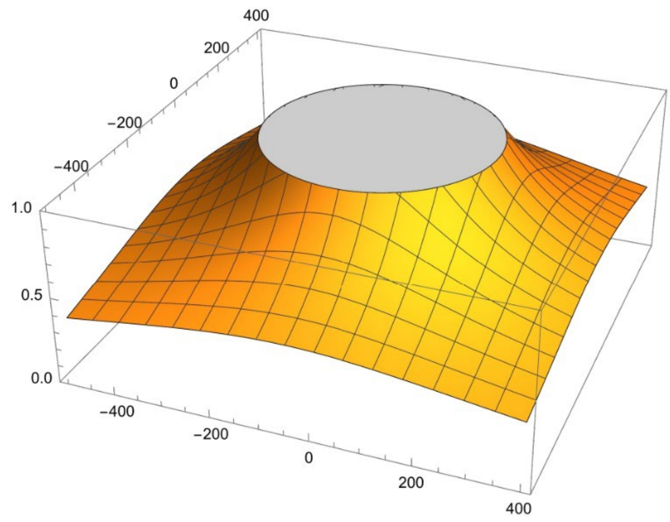


Figure 1. Stability surface for a strange, fixed point $x = 1$. Region of the complex plane: $[-450, 400] \times [-450,400]$.

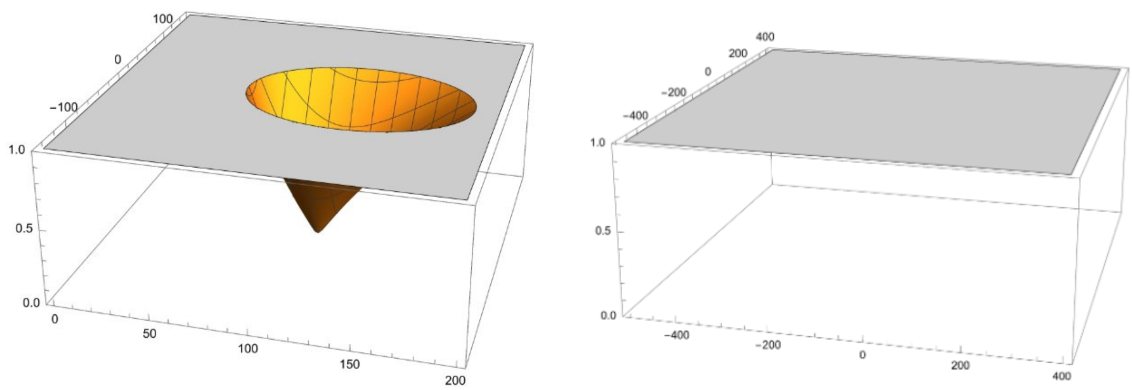


Figure 2. Stability surfaces for strange, fixed points $ex_i(\alpha)$ with $i = 1,2,\cdots,14$. Región of the complex plane: $[0, 200] \times [-150,150]$ and $[-450, 400] \times [-450,400]$.

In the following result, an analysis of the critical points of the rational operator is carried out.

Proposition 3. *The critical points of the rational operator are* $x = 0$ *and* $x = \infty$*, while the free critical points are* $x = -1$ *and the roots of the polynomial*

$$h(t) = 150 + 30\alpha + (1155 + 133)t + (3908 + 192)t^2 + (7489 - 9\alpha)t^3 + (9276 - 92\alpha)t^4 \\ +(7489 - 9\alpha)t^5 + (3908 + 192)t^6 + (1155 + 133)t^7 + (150 + 30\alpha)t^8,$$

and the roots of the polynomial, which are denoted by $cr_i(\alpha)$ *for* $i = 1,2,\cdots,8$.

The parameter space is constructed by meshing the complex plane, where each point corresponds to a value of the parameter $\alpha \in \mathbb{C}$. For each value α, the free critical point $cr_i(\alpha)$ is taken as the initial guess. The mesh point is colored red if the method converges to a root; otherwise, it is colored black. To draw the parameter planes, a mesh of 1000×1000 points, a tolerance of 10^{-3}, and a maximum of 100 iterations have been considered. To construct the parameter planes, the code proposed by Chicharro et al. (2013) is used.

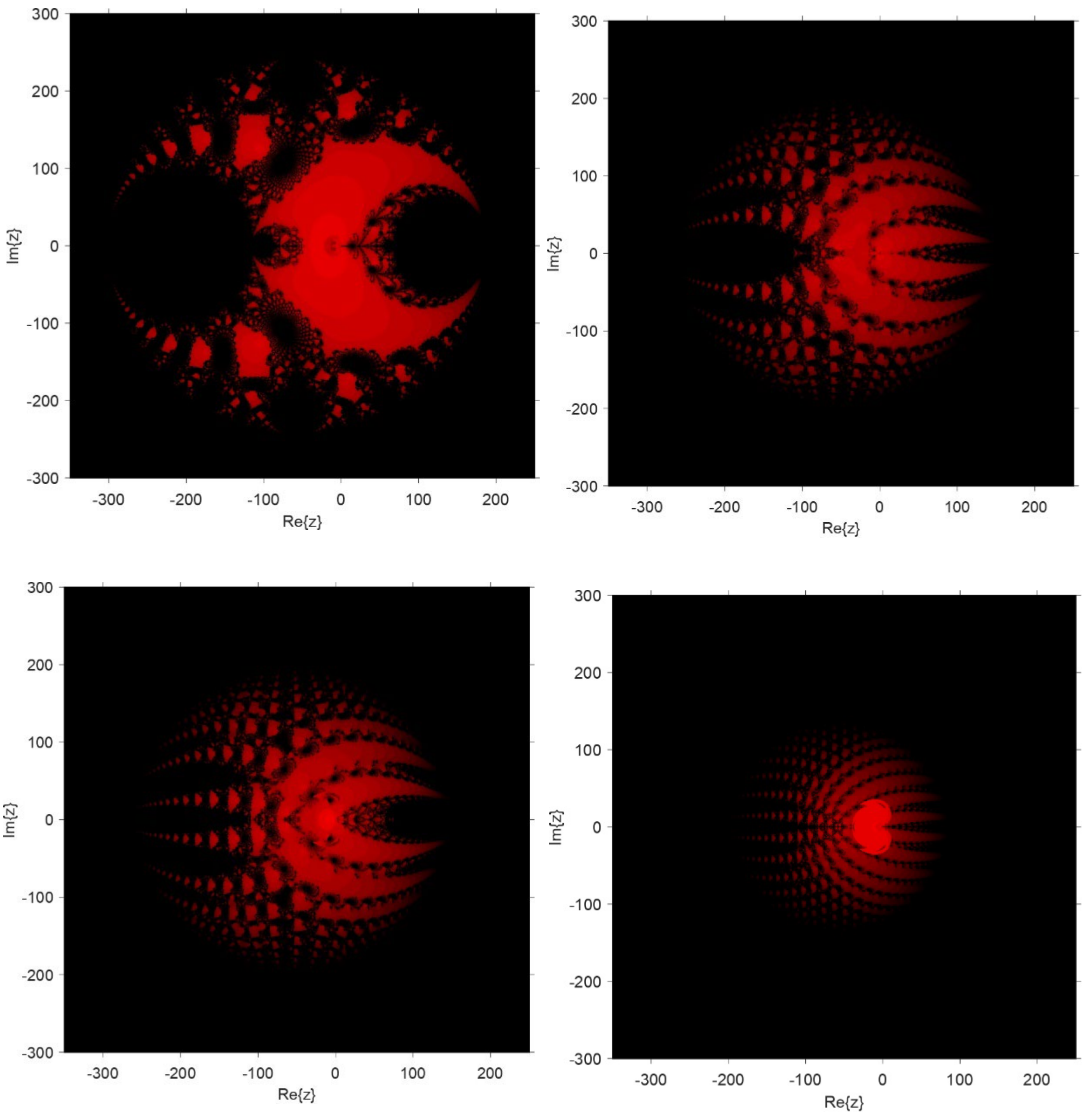


Figure 3. Parameter planes for the free critical points. Mesh size: 1000×1000; tolerance: 10^{-3}; maximum iterations: 100; color map: red (convergent), black (non-convergent); region of the complex plane: $[-350, 250] \times [-300,300]$.

The dynamical plane is a tool from complex dynamics that provides additional information beyond that obtained from stability surfaces and parameter planes. The dynamical plane is constructed similarly to the parameter plane: a grid of the complex plane is considered, and each point $x_0 \in \mathbb{C}$ represents an initial estimate for the iterative method. Depending on the convergence behavior, each point is represented by a specific color. For constructing the dynamical planes, the code proposed by Chicharro et al. (2013) is used. For the construction of the dynamical planes, values of the parameter α belonging to stable and unstable regions in the parameter planes are selected. A mesh of 1000×1000 points, a maximum of 100 iterations, and a tolerance of 10^{-3} are used.

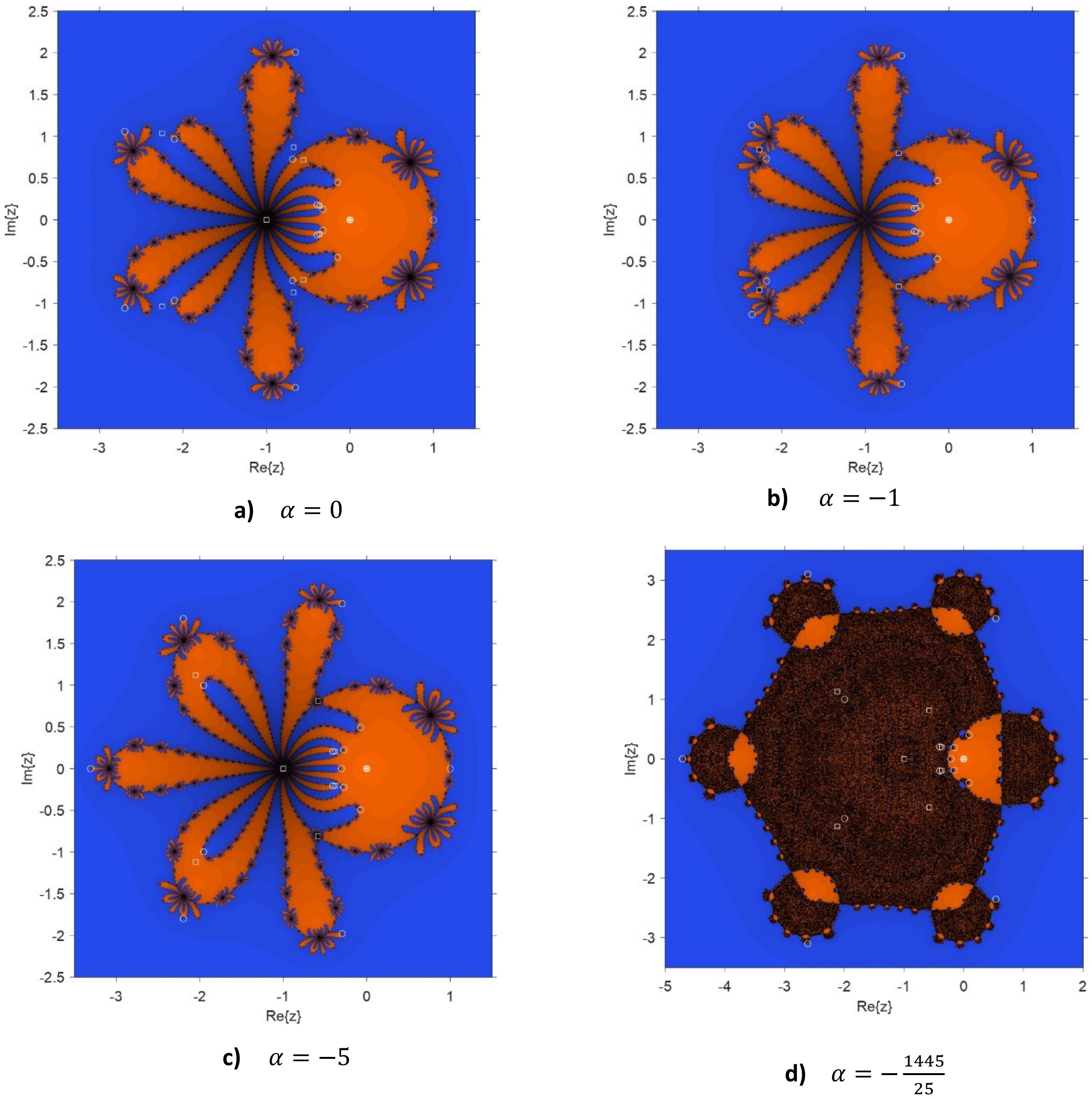
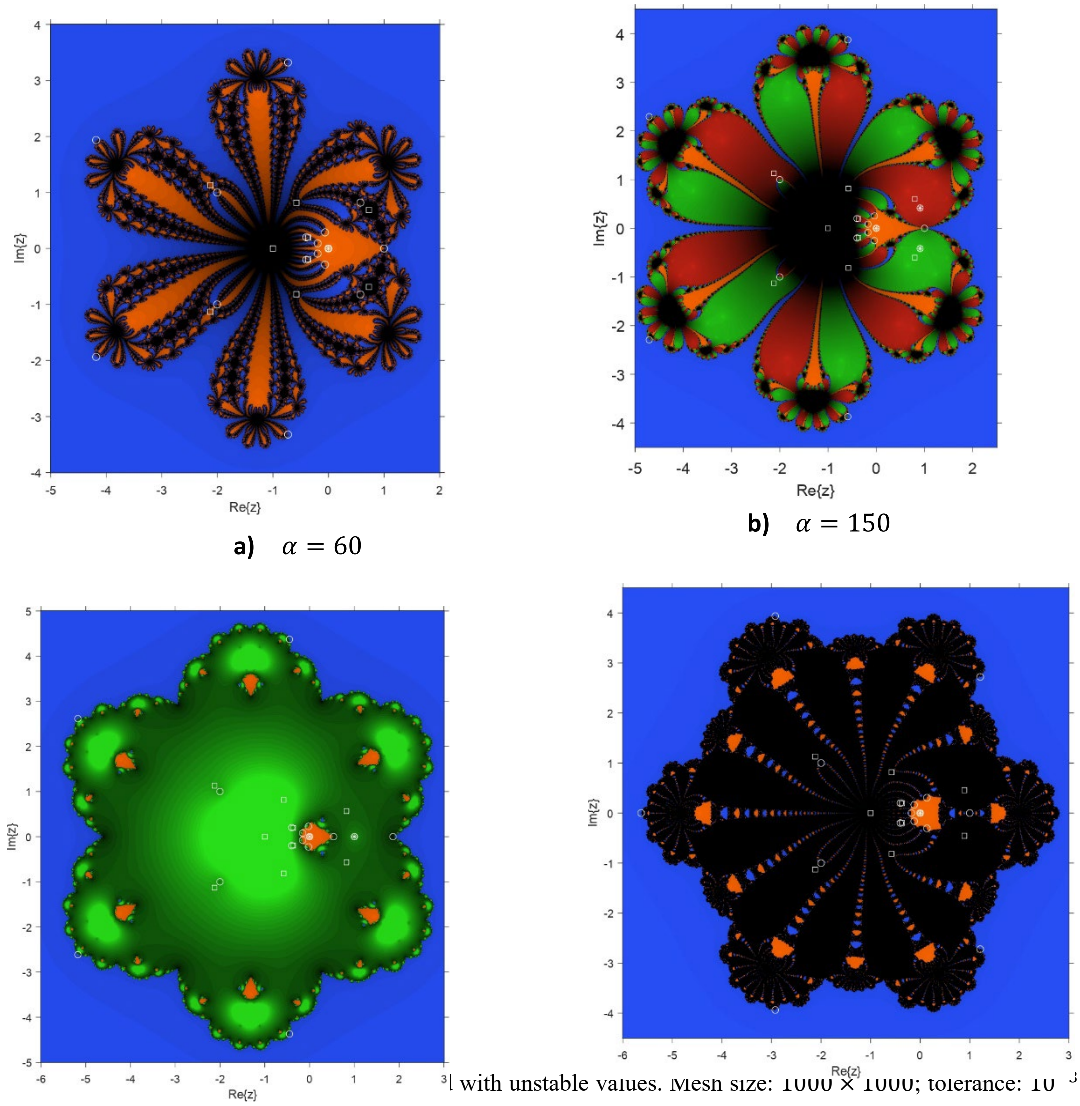


Figure 4. Dynamical planes associated with stable values. Mesh size: 1000×1000; tolerance: 10^{-3}; maximum iterations: 100.

In Figure 4, the values $\alpha = 0, -1, -5$, and $-\frac{1445}{25}$ have been considered. It can be observed that for $\alpha = 0, -1$, and -5, two basins of attraction appear: that of $x = 0$, represented in orange, and that of $x = \infty$ represented in blue, thus yielding a simple dynamic behavior for these parameters. However, when $\alpha = -\frac{1445}{25}$, black regions appear, which may indicate convergence issues of the method, even though α lies within a convergence region in the parameter planes.



a) $\alpha = 60$ b) $\alpha = 150$

c) $\alpha = 300$ d) $\alpha = -200$

with unstable values. Mesh size: 1000×1000; tolerance: 10^{-3}; maximum iterations: 100.

In Figure 5, when taking $\alpha = 60$, three basins of attraction can be observed: that of $x = 0$, that of $x = \infty$ and a black region indicating non-convergent zones. For $\alpha = 150$, four basins of attraction appear: the green one, which is associated with the strange fixed point $x = 1$, and the black one, which indicates slow convergence. When $\alpha = 300$, the green region becomes considerably larger and is associated with the strange fixed point $x = 1$. For $\alpha = -200$, black regions are observed, indicating that the method is not stable for this parameter.

To quantify the regions observed in the dynamical planes, a color-based analysis was performed on the inner frame of each figure. The tones associated with convergence (blue and orange) were grouped together, while the remaining colors (including green, red, and black) were classified as non-convergent regions. This provides an approximate measure of the percentage of mesh points that converge to a root versus those that diverge or remain undefined.

Table 1. Percentage of mesh classified by convergence.

α	Convergence (%)	Non-convergence (%)
60	71.47	28.53
150	40.60	59.40
-200	36.98	63.02
300	27.76	72.24

When considering the value of $\alpha = 60$, it is observed that 71.47% of the grid points converge to the roots of the polynomial. However, when $\alpha = 150, -200, 300$, there is a higher percentage of initial grid points for which the method does not converge. This is due to the presence of other basins of attraction associated with the strange fixed points.

4. Discussion

In this section, several numerical tests will be carried out to confirm the validity of the results related to the convergence and stability of the family M6(α). To perform the numerical tests, one value of the parameter α that generates a stable method and another value of α that generates an unstable method will be considered, and each method will be applied to ten nonlinear equations. The expressions of these equations and their respective roots are shown in Table 2.

MATLAB 2022b is used for numerical tests, with variable-precision arithmetic with 500 mantissa digits. The stopping criterion is established as $|f(x_{k+1})| < 10^{-100}$ or $|x_{k+1} - x_k| < 10^{-100}$, with a maximum number of 30 iterations. The approximate computational order of convergence (ACOC) will be obtained to verify the theoretical order of convergence. If the method does not converge within 30 iterations, the ACOC is indicated as “nc”, and if the ACOC fails to stabilize, the symbol “-” is used.

Table 2. Nonlinear test equations.

Nonlinear equations	Roots
$f_1(x) = \cos(x) - 2x^2 + 1 = 0$	$\xi \approx 0.90036$
$f_2(x) = 2 + (x-1)^3 + e^{x^2} = 0$	$\xi \approx -0.48322$
$f_3(x) = xe^x + \cos(x) - 3x^2 - 2 = 0$	$\xi \approx 1.95874$
$f_4(x) = \arctan(x^3) + \sqrt{x^2 + 2x} - 4 = 0$	$\xi \approx 1.78793$
$f_5(x) = \log(x^2 + 2) - x^3 + 1 = 0$	$\xi \approx 1.32435$

For the numerical experiments, the parameters $\alpha = 0$ and $\alpha = 300$ are chosen, where the former simplifies the rational operator and also provides a simple dynamical behavior, while the latter, according to Figure 5, exhibits an unstable behavior.

Table 3. Numerical results of M6(0) on nonlinear equations (1/2).

Function	x_0	$\lvert x_{k+1} - x_k\rvert$	$\lvert f(x_{k+1})\rvert$	Iterations	ACOC	Time
	$x_0 \approx \xi$					
f_1	0.8	1.0243×10^{-34}	5.3686×10^{-204}	3	6.0634	0.06392
f_2	-0.5	8.5634×10^{-66}	1.5786×10^{-390}	3	5.9952	0.09636
f_3	1.9	2.1979×10^{-29}	8.9641×10^{-170}	3	6.0769	0.07919
f_4	1.7	2.6226×10^{-55}	1.1284×10^{-330}	3	6.0345	0.09936
f_5	1.3	4.5353×10^{-51}	4.2986×10^{-301}	3	6.0119	0.07426
	$x_0 \approx 10\xi$					

Table 4. Numerical results of M6(0) on nonlinear equations (2/2).

Function	x_0	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	Iterations	ACOC	Time
f_1	8.0	1.6369×10^{-66}	8.9396×10^{-395}	5	5.9955	0.08241
f_2	-5.0	1.5400×10^{-34}	2.1337×10^{-198}	15	5.9537	0.31003
f_3	19.0	7.9402×10^{-93}	0	14	5.9996	0.25701
f_4	17.0	2.0179×10^{-73}	2.3415×10^{-439}	4	6.0089	0.11272
f_5	13.0	1.6049×10^{-28}	8.4429×10^{-166}	6	5.9010	0.11082

With $\alpha = 0$, the iterative family shows a highly stable and accurate performance, confirming an approximate convergence order of six in all analyzed cases. For initial conditions close to the root, the method converges in only three iterations, with minimal errors and residuals, demonstrating high numerical precision and low computational cost. Even when the initial point is farther from the root, the method preserves its stability and the same convergence order. However, it requires more iterations (4-15) and slightly longer execution times. Overall, the results confirm that the value $\alpha = 0$ provides an optimal performance within the family, combining efficiency, speed, and robustness with respect to variations in the initial condition.

Table 5. Numerical results of M6(300) on nonlinear equations.

Function	x_0	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	Iterations	ACOC	Time
	$x_0 \approx \xi$					
f_1	0.8	7.4561×10^{-22}	4.7700×10^{-125}	3	6.1290	0.06504
f_2	-0.5	1.4314×10^{-53}	1.9739×10^{-315}	3	5.9922	0.09011
f_3	1.9	2.5029×10^{-89}	0	4	5.9997	0.09704
f_4	1.7	7.3766×10^{-40}	9.9462×10^{-236}	3	6.0392	0.09768
f_5	1.3	3.0434×10^{-38}	2.5510×10^{-222}	3	6.0206	0.06722
	$x_0 \approx 10\xi$					
f_1	8.0	nc	nc	nc	nc	nc
f_2	-5.0	nc	nc	nc	nc	nc
f_3	19.0	nc	nc	nc	nc	nc
f_4	17.0	7.6319×10^{-74}	1.2198×10^{-439}	4	5.9973	0.12416
f_5	13.0	nc	nc	nc	nc	nc

When comparing the results obtained for $\alpha = 300$ with those for $\alpha = 0$, it is observed that the iterative family M6(α) maintains an approximate convergence order of six and high numerical accuracy in both cases when the initial point is close to the root . For $\alpha = 300$, the method solves in three or four iterations, with errors and residuals on the order of $10^{-22} - 10^{-89}$ and $10^{-125} - 10^{-315}$, respectively, values comparable to those obtained with $\alpha = 0$. However, when analyzing more distant initial conditions, the behavior differs significantly: while for $\alpha = 0$ the method remains convergent in all cases, for $\alpha = 300$ the process diverges for most of the test functions, except for one case where it stabilizes and preserves the theoretical order. Consequently, although both parameter values yield efficient performance near the root, $\alpha = 0$ provides greater global stability, whereas $\alpha = 300$ exhibits a more restrictive and unstable dynamic when the initial condition varies.

Next, a comparative analysis will be carried out between a stable method of the family M6(α) and five iterative methods of different orders, to evaluate the numerical performance in solving nonlinear equations. The parameter $\alpha = -5$ is chosen, since a more simplified rational operator is obtained, which is

$$O_{-5}(z) = \frac{z^7(z^2 + 4z + 5)(z^7 + 9z^6 + 37z^5 + 92z^4 + 149z^3 + 153z^2 + 73z + 14)}{(5z^2 + 4z + 1)(1 + 9z + 37z^2 + 92z^3 + 149z^4 + 153z^5 + 73z^6 + 14z^7)}.$$

Thus reducing the number of strange fixed points and free critical points. Moreover, $\alpha = -5$ belongs to the stable region in the parameter planes, and for $\alpha = -5$, according to Proposition 2, the strange fixed point $x = 1$ is repulsive.

The iterative methods considered in this study are the following:

- Newton's method, denoted as NM, and its expression is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots$$

- The method of Ostrowski (1960), denoted as MO and defined as

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(x_k)}{f(x_k) - 2f(y_k)} \dfrac{f(y_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots \end{cases}$$

- The method of Jarratt (1969), denoted as MJ, and its expression is

$$\begin{cases} y_k = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{1}{2}\dfrac{f(x_k)}{f'(x_k)}\left(\dfrac{3f'(y_k) + f'(x_k)}{3f'(y_k) - f'(x_k)}\right), \quad k = 0,1,2\cdots \end{cases}$$

- The sixth-order method proposed (Cordero et al., 2021), denoted as CMT, and its expression is

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \dfrac{f(y_k)}{2f[x_k, y_k] - f'(x_k)}, \\ x_{k+1} = z_k - (\beta + (1+\beta)\mu_k + (1-\beta)\nu_k)\dfrac{f(z_k)}{f'(x_k)}, \quad k = 0,1,2,\cdots, \end{cases}$$

In Tables 6 and 7, an initial estimate close to the solution η will be taken to compare the results obtained.

Table 6. Numerical performance of iterative methods in nonlinear equations for $x_0 \approx \xi$ (1/2)

Function	Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iterations	ACOC	Time
f_1	$M6(-5)$	3.2293×10^{-41}	1.10674×10^{-244}	3	6.2137	0.05451
$x_0 = 0.8$	MN	1.7526×10^{-80}	7.0977×10^{-160}	7	2.0000	0.09936
	MO	1.3807×10^{-79}	2.5820×10^{-316}	4	4.0000	0.06377
	MJ	1.2735×10^{-79}	1.8607×10^{-316}	4	4.0000	0.12055
	CMT	7.1089×10^{-40}	1.3002×10^{-235}	3	6.0389	0.05087

Table 7. Numerical performance of iterative methods in nonlinear equations for $x_0 \approx \xi$. (2/2)

Function	Method	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	Iterations	ACOC	Time
f_2	$M6(-5)$	3.1867×10^{-73}	2.5776×10^{-436}	3	6.067	0.08132
$x_0 = -0.5$	MN	2.2370×10^{-88}	1.2994×10^{-175}	7	2.0000	0.07574
	MO	1.2311×10^{-86}	1.8457×10^{-344}	4	4.0000	0.05218
	MJ	3.2218×10^{-88}	7.1392×10^{-351}	4	4.0000	0.06462
	CMT	4.5277×10^{-44}	8.5943×10^{-261}	3	6.0180	0.06042
f_3	$M6(-5)$	2.3575×10^{-33}	1.17534×10^{-194}	3	6.2017	0.05984
$x_0 = 1.9$	MN	6.5178×10^{-70}	4.7680×10^{-138}	7	2.0000	0.07021
	MO	1.2196×10^{-74}	2.7409×10^{-295}	4	4.0000	0.08726
	MJ	1.6662×10^{-74}	9.6986×10^{-295}	4	4.0000	0.06974

	CMT	3.1448×10^{-36}	1.0087×10^{-211}	3	6.0384	0.08516
f_4	M6(−5)	9.3647×10^{-54}	4.5613×10^{-321}	3	6.0247	0.08440
$x_0 = 1.7$	MN	9.7673×10^{-53}	3.1751×10^{-105}	6	2.0000	0.13373
	MO	5.4549×10^{-98}	4.0911×10^{-391}	4	4.0000	0.07875
	MJ	7.8331×10^{-26}	9.6803×10^{-103}	3	4.0208	0.07885
	CMT	6.9953×10^{-55}	5.5267×10^{-328}	3	6.0235	0.08172
f_5	M6(−5)	1.1450×10^{-57}	7.4011×10^{-342}	3	6.0454	0.06332
$x_0 = 1.3$	MN	4.6719×10^{-54}	8.6338×10^{-107}	6	2.0000	0.07306
	MO	3.4861×10^{-28}	3.0303×10^{-110}	3	4.0043	0.05021
	MJ	3.4529×10^{-28}	2.9109×10^{-110}	4	4.0043	0.06760
	CMT	4.4257×10^{-57}	5.4482×10^{-338}	3	6.0028	0.05720

In Tables 6 and 7, the initial approximation $x_0 \approx \eta$ has been considered. It is observed that Newton's method requires the largest number of iterations for all equations, which was expected due to its quadratic order of convergence. Jarratt's and Ostrowski's methods exhibit similar behavior with respect to the number of iterations. The M6(−5) performs three iterations for all equations, as do the other seventh-order methods. Furthermore, the sixth-order methods, including M6(−5), yield similar results for the values of $|x_{k+1} - x_k|$ and $|f(x_{k+1})|$.

Table 8. Numerical performance of iterative methods in nonlinear equations for $x_0 \approx 10\xi$. (1/2)

Function	Method	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	Iterations	ACOC	Time
f_1	M6(−5)	1.2871×10^{-95}	4.3603×10^{-508}	5	6.0085	0.08636
$x_0 = 8.0$	MN	2.9167×10^{-59}	1.9657×10^{-117}	10	2.0000	0.10010
	MO	2.3390×10^{-29}	2.1262×10^{-115}	5	3.9947	0.07894
	MJ	1.6725×10^{-31}	5.5358×10^{-124}	5	3.9964	0.09117
	CMT	1.4919×10^{-17}	1.1107×10^{-101}	4	5.5594	0.06457
f_2	M6(−5)	3.4182×10^{-54}	3.0512×10^{-317}	14	5.9126	0.32615
$x_0 = -5.0$	MN	2.8569×10^{-68}	8.9501×10^{-134}	22	2.0000	0.31122
	MO	4.4511×10^{-55}	1.7146×10^{-215}	10	4.0000	0.27329
	MJ	2.8811×10^{-46}	3.1665×10^{-180}	10	3.9998	0.24388
	CMT	1.6478×10^{-53}	3.3397×10^{-313}	9	5.9922	0.27125

Table 9. Numerical performance of iterative methods in nonlinear equations for $x_0 \approx 10\xi$. (2/2)

Function	Method	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	Iterations	ACOC	Time
f_3	M6(−5)	2.5266×10^{-99}	0	13	5.9980	0.29081
$x_0 = 19.0$	MN	4.2005×10^{-67}	1.9803×10^{-132}	28	2.0000	0.28778
	MO	5.3344×10^{-78}	1.0032×10^{-308}	13	4.0000	0.24844
	MJ	6.3608×10^{-65}	2.0597×10^{-256}	13	4.000	0.26074
	CMT	4.3124×10^{-23}	6.7067×10^{-133}	11	5.8435	0.22709
f_4	M6(−5)	3.4438×10^{-71}	1.1281×10^{-425}	4	6.0066	0.12779
$x_0 = 17.0$	MN	2.2233×10^{-67}	1.6452×10^{-134}	8	2.0000	0.13033
	MO	2.9920×10^{-91}	3.7029×10^{-364}	5	4.0000	0.12303
	MJ	5.0866×10^{-77}	1.7214×10^{-307}	5	4.0000	0.12738
	CMT	1.6918×10^{-72}	1.1057×10^{-433}	4	6.0061	0.12369
f_5	M6(−5)	2.3562×10^{-53}	5.6216×10^{-316}	6	5.9305	0.15132
$x_0 = 13.0$	MN	6.9556×10^{-56}	1.9137×10^{-110}	13	2.0000	0.14343
	MO	7.7210×10^{-73}	7.2918×10^{-289}	7	4.0000	0.12614
	MJ	7.2577×10^{-73}	5.6821×10^{-289}	7	4.0000	0.16738
	CMT	2.4872×10^{-67}	1.7165×10^{-399}	6	5.9970	0.12466

The results show that the method M6(−5) stands out for achieving high levels of numerical accuracy in just a few iterations. In all cases, it attains residuals close to zero and extremely small differences between iterations (up to 10^{-99}), which indicates high stability and an excellent approximation to the root. Moreover, the obtained ACOC remains close to 6 in all experiments, thus confirming the theoretical order of convergence of the method.

In the previous tables, a comparison of the methods has been made in terms of ACOC, errors, and execution time. Now, the Ostrowski efficiency index (Ostrowski, 1960) will be used, which is defined as

$$EI = p^{1/d},$$

where p is the order of convergence of the method and d is the number of functional evaluations performed. The Ostrowski efficiency index is useful because it helps avoid artificial accelerations in iterative methods. It has been demonstrated that the family M6(α) has a convergence order of $p = 6$ and involves a total of four functional evaluations — three evaluations of the function f at the points x_k, y_k, and z_k, and one evaluation of its derivative f'at the point x_k. Therefore, based on the above information, the Ostrowski efficiency index is

$$EI = 6^{1/4} \approx 1.5650845801.$$

Table 10 presents a comparison of the Ostrowski efficiency index of the family M6(α) with the Newton, Ostrowski, Jarratt, and CMT methods.

Table 10. Comparison of the efficiency index.

Method	Functional evaluations (d)	Order (p)	EI
M6(α)	$4\ (3\,f, 1\,f')$	6	$6^{1/4} \approx 1.5651$
MN	$2\ (1\,f, 1\,f')$	2	$2^{1/2} \approx 1.4142$
MO	$3\ (2\,f, 1\,f')$	4	$4^{1/3} \approx 1.5874$
MJ	$3\ (2\,f, 1\,f')$	4	$4^{1/3} \approx 1.5874$
CMT	$4\ (3\,f, 1\,f')$	6	$6^{1/4} \approx 1.5651$

The Kung–Traub conjecture (Kung & Traub, 1974) states that the convergence order of a memoryless iterative method with d functional evaluations per iteration is less than or equal to 2^{d-1} . When this bound is reached, the method is called optimal. Therefore, the theoretical efficiency frontier for the M6(α) family is $2^{4-1} = 8$; since this does not match the actual convergence order $p = 6$, the family is not optimal.

When compared with the previous methods, the M6(α) family proves to be more efficient than Newton's method and exhibits the same efficiency index (EI) as the CMT method. However, the Ostrowski and Jarratt methods have a slightly higher EI, which is to be expected since they are optimal methods. Although the proposed family is not optimal according to the Kung–Traub criterion, its convergence and stability properties make it a competitive option for high-precision computations.

5. Conclusions

In this work, a new iterative family has been developed to solve nonlinear equations. This family was constructed from the fourth-order uniparametric scheme proposed by Artidiello (2014), to which a third step based on the work of Moscoso-Martínez et al. (2023) was added. It was demonstrated that the new family for solving nonlinear equations has an order of convergence of 6; thus, the third step increases the convergence order by 2.

The research results inform the design of high-order iterative schemes, demonstrating that incorporating an adaptive third step can increase the order of convergence without significantly increasing computational cost. This reinforces the relevance of parameterized families for the construction and analysis of new rational operators, particularly through the combined use of tools from complex dynamics.

In the dynamic analysis of the family for solving the nonlinear equation, the rational operator was obtained using the polynomial $\boldsymbol{p(x) = (x - a)(x - b)}$ and a Möbius transformation, which made it possible to determine its fixed and critical points. Through the parameter planes, the regions of the complex plane that define the values of the parameter α for which the methods of the family are stable have been identified. In the dynamical planes, the different basins of attraction are shown, with bold italic alpha values that generate both stable and unstable methods.

Numerical experiments confirm that, for specific parameter values, the proposed family outperforms classical methods such as Newton, Ostrowski, Jarratt, and CMT in both efficiency and accuracy, while maintaining a high order of convergence even under initial conditions far from the root. Overall, the results demonstrate the robustness, stability, and applicability of the new iterative family, thereby consolidating it as an effective tool for the numerical analysis of nonlinear equations. The proposed family achieves an appropriate balance between accuracy and efficiency, as indicated by the Ostrowski efficiency index (EI ≈ 1.565) and the reduced number of functional evaluations per iteration. Moreover, the robustness analysis based on mesh-grid convergence percentages supports its practical adoption in scientific and engineering problems that require global stability and rapid convergence.

The iterative family was designed for solving nonlinear equations; however, it can be extended to the multidimensional case by adopting the corresponding notation. Therefore, one future research direction is to determine whether the order of convergence is preserved in the multidimensional setting. Additionally, a stability analysis could be performed using tools from fundamental dynamics, such as parameter lines and dynamical planes in the real plane.

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